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Reaction-Cattaneo systems with fluctuating relaxation time

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A reaction-Cattaneo equation with fluctuating relaxation time of the diffusive flux has been explored. It has been shown that depending on the strength of fluctuations, the dynamical system exhibits new oscillatory solutions as a result of Hopf and double Hopf bifurcations leading to spatiotemporal patterns. This analysis has been applied to two model nonlinear systems.

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I. INTRODUCTION

The reaction-diffusion systems are ubiquitous in natural sciences for understanding a wide class of far-from-equilibrium phenomena in spatially extended nonlinear dynamical systems [1–4]. Examples include, traveling waves, stationary, and nonstationary patterns, spirals, targets among many others. A reaction-diffusion equation describes the spatiotemporal evolution of dispersal of particles or species which are undergoing reactions among themselves. While the reactions follow the laws of chemical kinetics, the dispersal is governed by the diffusion equation. Although the standard diffusion equation lies at the heart of Brownian dynamics or stochastic processes, at large, it, however, suffers from a pathological pitfall. A close look at the fundamental Gaussian solution of the diffusion equation with a point source of the particles at distance $r=0$ at initial time $t=0$ immediately asserts that, there exists always a finite nonzero density of particles at very large distance even when the time is very small. The implication is that the particles have infinite speed. This unrealistic feature owes its origin to the lack of inertia of the particles since the successive time intervals associated with a diffusing particle are uncorrelated. Furthermore, this also makes it impossible to follow the motion of the particles when the time is short. To bypass this difficulty the model of correlated random walk was proposed by Furth and others [5–7]. At the macroscopic continuum level this corresponds to introducing nonzero relaxation time (τ) of flux by Cattaneo equation implying that the flux is delayed by a time τ with respect to the concentration gradient. The Cattaneo's modification of Fick's law [8–10] can be combined with reaction kinetics to obtain the reaction-telegraphic or reaction-Cattaneo equation, which can successfully overcome the above-mentioned difficulties. This equation has been studied over the last several years in a wide variety of contexts by several authors [10–13].

Before proceeding further we begin with a note that the delay time τ of the reaction-Cattaneo equation is closely related to correlated or persistent random walk. The persistence in time implies that a particle continues in its initial direction with a definite probability. Thus there is a close kinship between the delay time τ and the persistence of random walk as a matter of chance at the microscopic level. We may thus anticipate that when the system size is not too large the delay time is prone to stochastic fluctuations. This is particularly important when the chemical reactions are followed as elementary individual events as considered, for example, in Gillespie algorithm [14]. One thus expects that the delay time in the reaction-Catteneo equation is not a constant parameter but may undergo stochastic fluctuations around a positive mean for such systems or otherwise. A number of theoretical and experimental investigations have been made over the last several years to examine the effect of delay in several contexts [15–19]. For example, delay in nonlinear dynamical systems may change the stability boundaries, induce multi-rhythmicity, waves, chaos and pattern formation and other features of self-organization. The role of delay and its feedback is quite extensive and pervasive in chemistry, physics, and biology [20–23]. The use of delay differential equations in various systems such as the Oregonator model, Brown's model of Enzyme kinetics, solid state nucleation, mass transfer across membrane, etc., has been discussed by Roussel and Epstein [24–26]. Apart from these, one may also encounter situations where the constant delay does not describe realistic situations and the use of distributed delay [27–29] is advocated. While the majority of these work concern the kinetic terms of the reaction-diffusion system, the present work focuses on the intrinsic delay time which characterizes the time scale of relaxation of the diffusive flux due to the inertia of the diffusing particles. Thus our purpose in this paper is to introduce a reaction-Cattaneo equation with fluctuating delay and explore the associated transport-driven instabilities. We show that the resulting equation is characterized by additional nonlinearity and higher order time-derivatives. Linear stability analysis has been carried out to show that depending on the strength of fluctuating delay the homogeneous steady state of the dynamical system may undergo single Hopf or double Hopf bifurcations, exhibiting very rich complex dynamics leading to spatiotemporal patterns. The conditions for the bifurcations are applied to two nonlinear model systems.

II. REACTION-DIFFUSION EQUATION WITH FINITE MEMORY

We consider the concentration of a reacting species or field variable $u(r,t)$, a function of space (r) and time (t) in terms of a reaction-diffusion system. The reaction-diffusion

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equation can be constructed phenomenologically from the continuity equation with a source term $f(u)$

$$\frac{\partial u(r,t)}{\partial t} = -\frac{\partial J(r,t)}{\partial r} + f(u), \tag{2.1}$$

where $J(r,t)$ is the flux of $u(r,t)$ and Fick's law

$$J(r,t) = -D\frac{\partial u(r,t)}{\partial r} \tag{2.2}$$

to obtain

$$\frac{\partial u(r,t)}{\partial t} = D\frac{\partial^2 u(r,t)}{\partial r^2} + f(u). \tag{2.3}$$

Here D refers to the diffusion coefficient of the species. A simple generalization of the reaction-diffusion equation is to include the effect of finite memory transport with Cattaneo's modification of Fick's law in the form [9,10]

$$J(r,t+\tau) = -D\frac{\partial u(r,t)}{\partial r} \tag{2.4}$$

Equation (2.4) implies that a concentration gradient at a time t causes a flux at a later time $(t+\tau)$, where τ is the delay time of the particles in adjusting one definite direction of motion. Making an expansion of J in Eq. (2.4) up to first order in τ in a Taylor series and differentiating the resulting equation with respect to r one obtains

$$\frac{\partial J(r,t)}{\partial r} + \tau\frac{\partial^2 J(r,t)}{\partial r\,\partial t} = -D\frac{\partial^2 u(r,t)}{\partial r^2}. \tag{2.5}$$

On the other hand differentiation of Eq. (2.1) with respect to t yields

$$\frac{\partial^2 u(r,t)}{\partial t^2} = -\frac{\partial^2 J(r,t)}{\partial r\,\partial t} + f'(u)\frac{\partial u(r,t)}{\partial t}. \tag{2.6}$$

Elimination of $J(r,t)$ from Eq. (2.5) and (2.6) with the help of Eq. (2.1) results in the following reaction-telegraph equation, a hyperbolic equation of the form

$$\tau\frac{\partial^2 u(r,t)}{\partial t^2} + [1-\tau f'(u)]\frac{\partial u(r,t)}{\partial t} = D\frac{\partial^2 u(r,t)}{\partial r^2} + f(u). \tag{2.7}$$

As $\tau \rightarrow 0$, the reaction-telegraph equation reduces to usual parabolic differential equation, i.e., ordinary reaction-diffusion equation. Thus a generic difference arises when τ is small but finite. In general, the smallness is ensured by the condition $\tau < \frac{1}{|f'(u)|}$, where $\frac{1}{|f'(u)|}$ is a measure of reaction time scale [8]. It may, however, be pointed out that linear stability analysis in certain cases may be carried out without this approximation. We specifically refer to [30–32]. Equation (2.7) is the reaction-Cattaneo equation with constant delay τ and is the basis for further analysis.

III. REACTION-CATTANEO SYSTEM WITH FLUCTUATING DELAY

Our purpose in this section is to explore the dynamics of reaction-Cattaneo system with fluctuating delay time $\tau(t)$. To this end we now assume that the delay time (τ) fluctuates around a positive mean value τ_0 such that

$$\tau(t) = \tau_0 + \eta(t), \tag{3.1}$$

where the noise term $\eta(t)$ is assumed to be Gaussian with zero mean

$$\langle \eta(t) \rangle = 0 \tag{3.2}$$

and δ correlated as

$$\langle \eta(t)\eta(t') \rangle = \epsilon\delta(t-t'), \tag{3.3}$$

ϵ is the noise strength of the fluctuation. We assume further that ϵ is very small which implies that the width of Gaussian distribution is very narrow. Ideally the fluctuating delay at times may have negative values. But the narrowness of the width around a relatively large positive mean ensures that the probability of such events is overwhelmingly small so that unphysical situations do not arise at the macroscopic limit. Using Eq. (3.1) in Eq. (2.7) and making an ensemble average we obtain

$$\tau_0\left\langle \frac{\partial^2 u}{\partial t^2} \right\rangle + \left\langle \frac{\partial u}{\partial t} \right\rangle - \tau_0\left\langle f'(u)\frac{\partial u}{\partial t} \right\rangle + \left\langle \eta(t)\frac{\partial^2 u}{\partial t^2} \right\rangle - \left\langle \eta(t)f'(u)\frac{\partial u}{\partial t} \right\rangle = D\left\langle \frac{\partial^2 u}{\partial x^2} \right\rangle + \langle f(u) \rangle. \tag{3.4}$$

The above equation can be put into a more tractable form by using Novikov's theorem [33] for Gaussian noise processes. According to this theorem the correlation function involving noise in multiplicative form can be written down as follows:

$$\langle f(u)\eta(t) \rangle = \epsilon \langle f(u)f'(u) \rangle, \tag{3.5}$$

where f is function of u and f' is the first derivative with respect to u. Applying Novikov's theorem [33] on the terms $\langle \eta(t)\frac{\partial^2 u}{\partial t^2} \rangle$ and $\langle \eta(t)f'(u)\frac{\partial u}{\partial t} \rangle$ it is easy to show

$$\left\langle \eta(t)\frac{\partial^2 (u)}{\partial t^2} \right\rangle = \epsilon\frac{\partial^4 \langle u \rangle}{\partial t^4} \tag{3.6}$$

and

$$\left\langle \eta(t)f'(u)\frac{\partial u}{\partial t} \right\rangle = \epsilon\left\langle f'(u)f''(u)\left(\frac{\partial u}{\partial t}\right)^2 \right\rangle. \tag{3.7}$$

Equation (3.4) can then be rewritten with the help of above two relations as follows:

$$\tau_0\frac{\partial^2 \langle u \rangle}{\partial t^2} + \frac{\partial \langle u \rangle}{\partial t} - \tau_0\left\langle f'(u)\frac{\partial u}{\partial t} \right\rangle + \epsilon\frac{\partial^4 \langle u \rangle}{\partial t^4} - \epsilon\left\langle f'(u)f''(u)\left(\frac{\partial u}{\partial t}\right)^2 \right\rangle = D\frac{\partial^2 \langle u \rangle}{\partial x^2} + \langle f(u) \rangle. \tag{3.8}$$

The above equation is complicated by the presence of several terms characterized by the strength of fluctuation of delay, ϵ. A further simplification can, however, be made at this stage by making use of mean field approximation [34] to write the average of the product as the product of averages so that the higher order correlations are neglected. We thus obtain

$$\epsilon\frac{\partial^4\langle u\rangle}{\partial t^4}+\tau_0\frac{\partial^2\langle u\rangle}{\partial t^2}+[1-\tau_0 f'(\langle u\rangle)]\frac{\partial\langle u\rangle}{\partial t}-\epsilon f'(\langle u\rangle)f''(\langle u\rangle)\left(\frac{\partial\langle u\rangle}{\partial t}\right)^2=D\frac{\partial^2\langle u\rangle}{\partial x^2}+f(\langle u\rangle). \tag{3.9}$$

In the limit $\epsilon\to 0$, Eq. (3.9) reduces to usual reaction-Cattaneo Eq. (2.7).

IV. TRANSPORT-DRIVEN INSTABILITIES

We now return to Eq. (3.8) and explore the delay-induced bifurcations of the reaction-diffusion system. The homogeneous steady states are the fixed points u_0 of the dynamical system defined as

$$f(u_0)=0. \tag{4.1}$$

The spatiotemporal perturbation $\delta u(x,t)$ on a homogeneous steady state is given by

$$u(x,t)=u_0+\delta u(x,t) \tag{4.2}$$

Linearizing Eq. (3.8) around the steady state u_0 we obtain the dynamics of spatiotemporal perturbation

$$\tau_0\frac{\partial^2}{\partial t^2}\langle\delta u\rangle+\frac{\partial}{\partial t}\langle\delta u\rangle-\tau_0 f'(u_0)\frac{\partial}{\partial t}\langle\delta u\rangle+\epsilon\frac{\partial^4}{\partial t^4}\langle\delta u\rangle=D\frac{\partial^2}{\partial x^2}\langle\delta u\rangle+f'(u_0)\langle\delta u\rangle. \tag{4.3}$$

Expressing spatiotemporal perturbation $\delta u(x,t)$ as $\delta u(x,t)=\delta u_0 e^{\lambda t}\cos kx$ and upon inserting it in Eq. (4.3) we obtain the following dispersion relation:

$$\epsilon\lambda^4+\tau_0\lambda^2+(1-\tau_0 f'(u_0))\lambda+[Dk^2-f'(u_0)]=0. \tag{4.4}$$

The dynamics of the spatiotemporal perturbation is determined by the nature of dispersion relation clearly reflecting the nature of stability of the homogeneous steady state. We begin by considering the system without delay in flux, so that Eq. (4.4) yields

$$\lambda=-[Dk^2-f'(u_0)]. \tag{4.5}$$

The system evolves from a linearly unstable steady state $[f'(u_0)>0]$ and the reaction-diffusion system finally settles down to a diffusion-driven stable state if $Dk^2>f'(u_0)$.

We now discuss the following two cases:

(i) For $\tau_0\neq 0$ and $\epsilon=0$, i.e., the system has a fixed delay in diffusion; Eq. (4.4) becomes

$$\tau_0\lambda^2+[1-\tau_0 f'(u_0)]\lambda+[Dk^2-f'(u_0)]=0, \tag{4.6}$$

where the two different roots of λ are

$$\lambda_\pm=\frac{1}{2\tau_0}\{-[1-\tau_0 f'(u_0)]\pm\sqrt{[1-\tau_0 f'(u_0)]^2-4\tau_0[Dk^2-f'(u_0)]}\}. \tag{4.7}$$

(a) If $[1-\tau_0 f'(u_0)]^2>4\tau_0[Dk^2-f'(u_0)]$ and $1>\tau_0 f'$ then λ-s are real, negative, i.e., the system settles down to a stable state.

(b) If $[1-\tau_0 f'(u_0)]^2>4\tau_0[Dk^2-f'(u_0)]$ and $1<\tau_0 f'$ then λ-s are always real, positive, i.e., the homogeneous steady state remains unstable, in presence of perturbation.

(c) If $[1-\tau_0 f'(u_0)]^2<4\tau_0[Dk^2-f'(u_0)]$ then λ-s are a pair of complex conjugates. The system oscillates in time and depending upon the sign of $-[1-\tau_0 f'(u_0)]$, the unstable or stable state is reached. When $\tau_0=\frac{1}{f'(u_0)}$ (since the state is linearly unstable with respect to homogeneous perturbation i.e., $f'(u_0)>0$, τ_0 is positive) and $\epsilon=0$, then Eq. (4.4) becomes

$$\lambda^2=-f'(u_0)[Dk^2-f'(u_0)]. \tag{4.8}$$

Here the roots are purely imaginary and therefore the system lies at the Hopf bifurcation boundary. Hence for this particular value of delay strength the system finally evolves to a purely oscillatory state. This implies that the constant delay gives rise to spatially inhomogeneous self-sustained temporal oscillations. As a result the space-time profile is expected to exhibit a coherent structure. We will illustrate these structures in terms of two examples in the next section.

(ii) In all the above-mentioned cases the delay is kept constant. However, when the delay fluctuates in time so that $\epsilon\neq 0$, we are led the following situation. Thus for $\tau_0\neq 0$, $\epsilon\neq 0$ but $\tau_0=\frac{1}{f'(u_0)}$ Eq. (4.4) becomes

$$f'(u_0)\epsilon\lambda^4+\lambda^2+f'(u_0)[Dk^2-f'(u_0)]=0. \tag{4.9}$$

This is a biquadratic equation and can be written in the form

$$A\lambda^4+B\lambda^2+C=0, \tag{4.10}$$

where

$$A=\epsilon f'(u_0),$$

$$B=1,$$

$$C=f'(u_0)[Dk^2-f'(u_0)]. \tag{4.11}$$

The two pairs of roots can be evaluated as

$$\lambda_\pm^2=\frac{1}{2\epsilon f'(u_0)}[-1\pm\sqrt{\{1-4\epsilon[f'(u_0)]^2[Dk^2-f'(u_0)]\}}]. \tag{4.12}$$

Now for $1>4\epsilon[f'(u_0)]^2[Dk^2-f'(u_0)]$, $\lambda_\pm^2<0$ and two pairs of imaginary roots with opposite sign develops. This is the condition for double Hopf bifurcation. The nature of dispersion relation suggests that the spatially inhomogeneous profiles oscillate in time because of the superposition of four complex exponentials for each wave vector. This results in the emergence of complex spatiotemporal patterns. These patterns undergo changes along with the variation of noise strength of delay as shown in the next section.

V. APPLICATIONS

A. Fisher equation

Let us consider first the case of Fisher equation. The source term in this equation represents a birth death process with logistic population growth according to $f(u)=ru(1-\frac{u}{K})$ where r is the linear reproduction rate and K is the carrying capacity of the environment. The resulting equation with D as diffusion coefficient, i.e.,

$$\frac{\partial u}{\partial t} = ru\left(1-\frac{u}{K}\right) + D\frac{\partial^2 u}{\partial x^2} \tag{5.1}$$

describes an one-dimensional model for the spread of advantageous gene in a population. The homogeneous steady states, the fixed points u_0 of the dynamical system are $u_0=0$, K of which $u_0=0$ is a linearly unstable fixed point.

Applying our previous analysis to this system it follows from Eq. (4.4)

$$\epsilon\lambda^4 + \tau_0\lambda^2 + (1-\tau_0 r)\lambda + (Dk^2 - r) = 0. \tag{5.2}$$

When $\tau_0=0$ and $\epsilon=0$

$$\lambda = r - Dk^2, \tag{5.3}$$

the one component reaction-diffusion system settles down to a diffusion-driven stable state for $Dk^2>r$.

When $\tau_0\neq 0$ and $\epsilon=0$, i.e., the system possesses a fixed delay time in flux, the characteristic equation takes the form

$$\tau_0\lambda^2 + (1-\tau_0 r)\lambda + (Dk^2 - r) = 0, \tag{5.4}$$

where

$$\lambda_\pm = \frac{1}{2\tau_0}\left[-(1-\tau_0 r) \pm \sqrt{(1-\tau_0 r)^2 - 4\tau_0(Dk^2-r)}\right]. \tag{5.5}$$

The subsequent three cases are as follows:

(i) If $(1-\tau_0 r)^2>4\tau_0(Dk^2-r)$ and $1>\tau_0 r$ then $\lambda_\pm$ are real and negative and the system settles down to a stable steady state.

(ii) If $(1-\tau_0 r)^2>4\tau_0(Dk^2-r)$ and $1<\tau_0 r$ then $\lambda_\pm$ are real, positive and the system becomes unstable with time.

(iii) If $(1-\tau_0 r)^2<4\tau_0(Dk^2-r)$ then $\lambda_\pm$ are complex conjugates and the system has a pair of oscillatory solutions and the stability depends on the sign of $(1-\tau_0 r)$. When τ_0 is varied the system passes through a Hopf bifurcation point at $\tau_0=1/r$. At this point Eq. (4.4) becomes

$$\lambda^2 + r(Dk^2 - r) = 0 \tag{5.6}$$

and λ becomes purely imaginary implying oscillatory solutions. These oscillatory solutions can be realized as spatiotemporal patterns. This is shown in Fig. 1. With increasing r the profile of the periodic temporal oscillation becomes more rapid over a range of spatially inhomogeneous distribution of concentration. It is thus apparent that when the constant time delay τ_0 matches the reaction time $1/r$ the Hopf bifurcation induces deterministic periodic oscillations whose spatial properties are determined by the growth rate of the logistic reaction term.

(iv) When $\tau_0\neq 0$, $\epsilon\neq 0$ but $\tau_0=\frac{1}{r}$ then Eq. (4.4) becomes

$$\epsilon r\lambda^4 + \lambda^2 + r(Dk^2 - r) = 0, \tag{5.7}$$

where

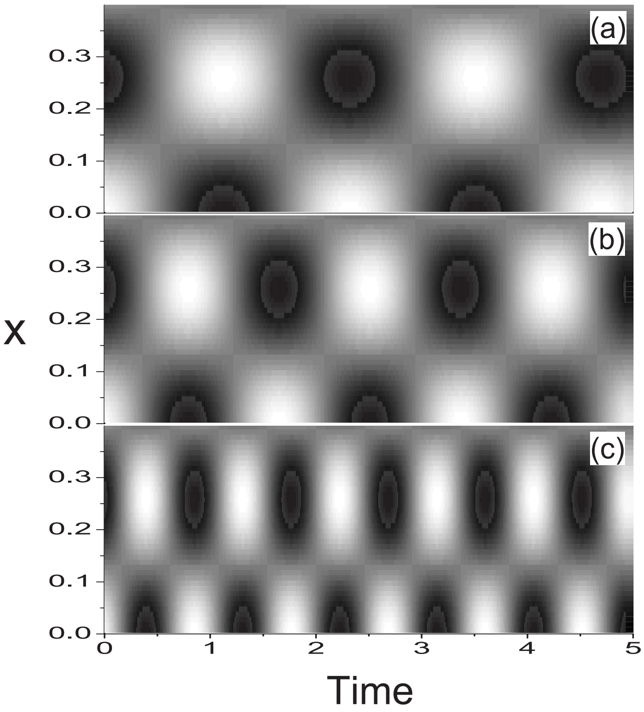

FIG. 1. The spatiotemporal patterns in Fisher equation due to Hopf bifurcation for constant delay $\tau_0=1/r$ for several values of r; (a) $r=0.5$, (b) $r=1.0$, and (c) $r=5.0$ and for $D=0.5$.

$$\lambda_\pm^2 = \frac{1}{2\epsilon r}\left\{-1 \pm \sqrt{[1-4\epsilon r^2(Dk^2-r)]}\right\}. \tag{5.8}$$

Under the condition $1>4\epsilon r^2(Dk^2-r)$, $\lambda_\pm^2<0$ and two pairs of imaginary roots with opposite sign develop. This special condition leads to double Hopf bifurcation in Fisher equation and the homogeneous state $u_0=0$ exhibits complex oscillations. These complex spatiotemporal oscillations are demonstrated in Fig. 2 for several values of noise strength ϵ for fixed values of r and D. By varying the noise strength of time delay the time period of oscillatory pattern can be changed. Or, in other words, with a proper choice of the constant part of the delay parameter τ_0 which matches the reaction time $1/r$, the noise strength imparts a significant effect on the nature of coherence of spatiotemporal dynamics.

B. Cubic autocatalytic reaction

A simple way to design a chemical oscillator is to look for systems that obey rate equations with cubic nonlinearity. The model proposed by Boissonade and De Kepper [35,36] has a generic form of a bistable system as follows:

$$\dot{u} = -(u^3 - \mu u + \gamma), \tag{5.9}$$

where the species u is produced autocatalytically (the μu term). The simple analysis of the nullcline shows that it can have one unstable and two stable branches for appropriate values of the parameters μ and γ. A prototypical cubic kinetics which has been extensively used [37,38] over a couple of decades in wave-front propagation problems in reaction-

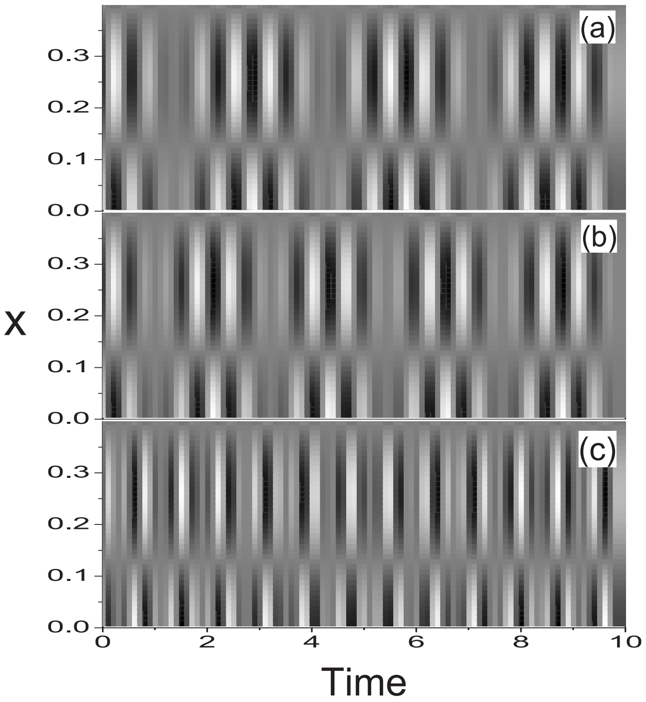

FIG. 2. The spatiotemporal patterns in Fisher equation due to double Hopf bifurcation for constant delay $\tau_0=1/r$ and for several values of noise strength (a) $\epsilon=5.0\times10^{-7}$, (b) $\epsilon=5.0\times10^{-5}$, and (c) $\epsilon=5.0\times10^{-3}$ of fluctuating delay time for $r=1.0$ and $D=0.5$.

diffusion systems is the model of iodate-arseneous acid reaction

$$\mathrm{IO_3}^- + 3\mathrm{H_3AsO_3} + 5\mathrm{I}^- \rightarrow 6\mathrm{I}^- + 3\mathrm{H_3AsO_4}. \quad (5.10)$$

The reaction is autocatalytic in iodide ion and the experimental rate equation takes the form

$$\frac{\partial u}{\partial t} = (k_1 + k_2 u)u(I_0 - u)h^2 = f(u), \quad (5.11)$$

where u denotes the concentration of iodide ion. h and I_0 are constant parameters which refer to the concentration of hydrogen ion and the initial concentration of iodate, respectively. k_1 and k_2 are the kinetic constants. The above equation has three steady states of the dynamics, viz.

$$u_0 = I_0, \quad -\frac{k_1}{k_2}, \quad 0. \quad (5.12)$$

The experimentally admissible parameters are given by, $k_1=4.5\times10^3\ \mathrm{M^{-3}\,s^{-1}}$; $k_2=1.0\times10^8\ \mathrm{M^{-4}\,s^{-1}}$; $D=2\times10^{-5}\ \mathrm{cm^2\,s^{-1}}$; $I_0=5.0\times10^{-3}$ M; $h=7.1\times10^{-3}$ M.

We apply our analysis of bifurcation theory to this system. Equation (4.4) takes the form

$$\epsilon\lambda^4 + \tau_0\lambda^2 + (1 - \tau_0 k_1 I_0 h^2)\lambda + (Dk^2 - k_1 I_0 h^2) = 0. \quad (5.13)$$

For $\tau_0=0$ and $\epsilon=0$ the system approaches to a diffusion-driven stable state for $k^2 > \frac{k_1 I_0 h^2}{D}$, i.e., for $k_c > 0.132$.

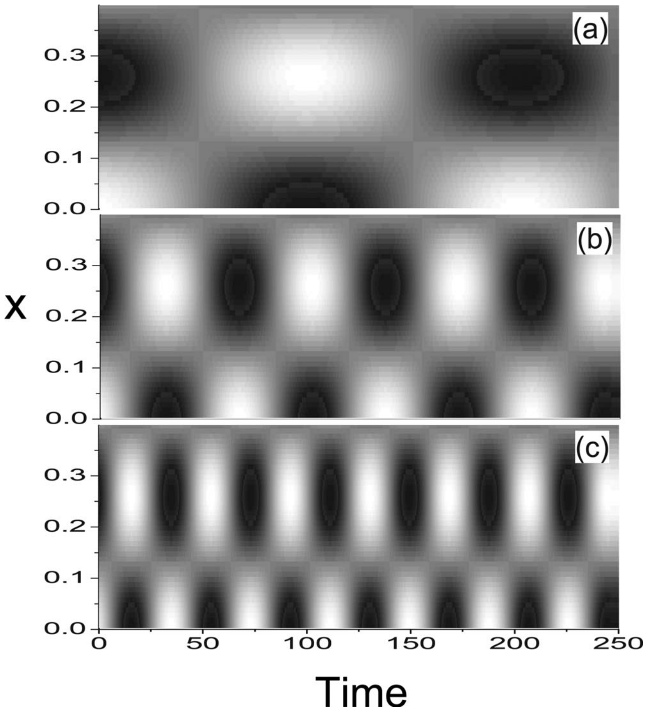

FIG. 3. The spatiotemporal patterns in iodate-arseneous acid reaction due to Hopf bifurcation for a constant delay $\tau_0=1/k_1I_0h^2$ and $D=2\times10^{-5}\ \mathrm{cm^2\,s^{-1}}$ for several values of hydrogen ion concentration (a) $h=1.0\times10^{-2}$ M, (b) $h=3.0\times10^{-2}$ M, and (c) $h=6.0\times10^{-2}$ M. The values of other parameters are mentioned in the text.

Now let us consider the case for which $\tau_0\neq0$ and $\epsilon=0$, i.e., the system with a fixed delay. Equation (4.16) is reduced to

$$\tau_0\lambda^2 + (1 - \tau_0 k_1 I_0 h^2)\lambda + (Dk^2 - k_1 I_0 h^2) = 0, \quad (5.14)$$

where

$$\lambda_\pm = \frac{1}{2\tau_0}\Big[-(1 - \tau_0 k_1 I_0 h^2) \pm \sqrt{(1 - \tau_0 k_1 I_0 h^2)^2 - 4\tau_0(Dk^2 - k_1 I_0 h^2)}\Big]. \quad (5.15)$$

Here if $\tau_0=\frac{1}{k_1 I_0 h^2}$ then the eigenvalues are imaginary with opposite sign. This corresponds to a Hopf bifurcation scenario. However, with fluctuating delay around a steady mean value of $\tau_0=\frac{1}{k_1 I_0 h^2}$ the characteristic equation becomes

$$\epsilon k_1 I_0 h^2 \lambda^4 + \lambda^2 + (Dk^2 - k_1 I_0 h^2)k_1 I_0 h^2 = 0, \quad (5.16)$$

$$\lambda_\pm^2 = \frac{1}{2\epsilon k_1 I_0 h^2}\Big[-1 \pm \sqrt{1 - 4\epsilon(k_1 I_0 h^2)^2(Dk^2 - k_1 I_0 h^2)}\Big]. \quad (5.17)$$

Now for the condition $1 > 4\epsilon(k_1 I_0 h^2)^2(Dk^2 - k_1 I_0 h^2)$ two pairs of imaginary roots develop. This corresponds to the condition of double Hopf bifurcation. As before both Hopf and double Hopf bifurcations are manifested in the spatiotemporal patterns shown in Figs. 3 and 4. With increase of h, i.e., hydrogen ion concentration, the periodicity in time becomes

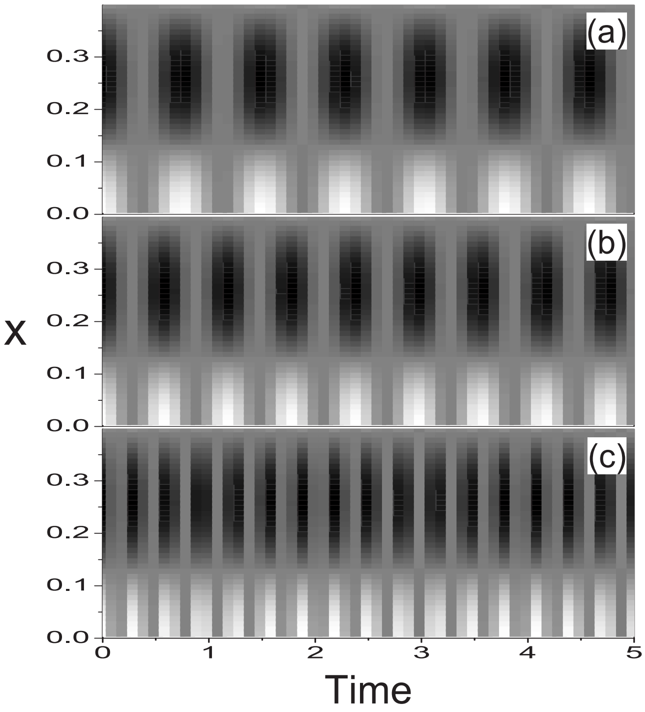

FIG. 4. The spatiotemporal patterns in iodate-arseneous acid reaction due to double Hopf bifurcations for constant delay $\tau_0 = 1/k_1 I_0 h^2$ and for several values of noise strength (a) $\epsilon = 5.0 \times 10^{-7}$, (b) $\epsilon = 5.0 \times 10^{-5}$, and (c) $\epsilon = 5.0 \times 10^{-3}$ of fluctuating delay time for $h = 7.0 \times 10^{-3}$ M and $D = 2 \times 10^{-5}$ cm^2 s^{-1}. The values of other parameters are mentioned in the text.

more and more prominent in the spatially inhomogeneous oscillations for an appropriate value of the constant delay (Fig. 3). The time period, in general, tends to be shortened when the delay fluctuates in time as a result of double Hopf bifurcation (Fig. 4). We close this section with a remark on the condition for controlling delay in the system. The fluctuation of delay time around the mean positive value depends on the concentration of $[IO_3^-]$ and $[H^+]$. Hence by manipulating the concentration of the above-mentioned species the value of the mean delay time for achieving the condition for the double Hopf bifurcation can be controlled. A simple arithmetic based on admissible concentrations of the iodate and $[H^+]$ shows the mean delay time to be 8.81×10^2 s which is too high for experimental purpose. However, the experimentally realizable delay time (8.8 s) can be obtained for relatively higher range 7.1×10^{-2} M of concentration of H^+.

VI. CONCLUSION

In this paper we have introduced a reaction-Cattaneo equation with fluctuating delay. At the microscopic level this takes care of the persistence or correlation of the underlying random walk when the system size is relatively small. On the other hand its macroscopic description corresponds equivalently to a relaxation time of the diffusive flux which fluctuates around a mean. This fluctuation in delay modifies the characteristics of the transport-driven instabilities of the dynamical system. It has been shown that depending on the strength of fluctuations the system admits of single and double Hopf bifurcations giving rise to emergence of new oscillating solutions leading to spatiotemporal patterns. The analysis has been applied to two well-known dynamical systems to illustrate these features in terms of realistic situations.

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